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Complex Reflection Subgroups of Real Reflection Groups

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We find all irreducible rank n complex reflection subgroups of finite irreducible rank $2n$ real reflection groups.

Key Words: real reflection groups; complex reflection groups.

1. INTRODUCTION

In this paper a triple (V, G, W) denotes an n-dimesnional $(n > 1)$ complex vector space V, an irreducible rank n truly complex (not complexified) reflection group G in V, and a rank $2n$ finite real reflection group W in ${}_{\mathbf{R}}V$, such that $G \subset W$ and W is minimal for G; i.e., there is no real reflection group W' such that $G < W' < W$. We classify such triples up to isomorphism (see Table 4), where two triples (V_1, G_1, W_1) and (V_2, G_2, W_2) are isomorphic if there exists a complex linear isomorphism ϕ: $V_1 \to V_2$ such that $\phi G_1 \phi^{-1} = G_2$ and $\phi W_1 \phi^{-1} = W_2$. Hereafter, we will suppress the vector space and write simply $G < W$ when referring to a triple (V, G, W).

If W is a real reflection group, Φ_W denotes its root system and $\mathrm{Aut}(\Phi_W)$ the orthogonal root system automorphisms. In [11] Springer defined *regular* elements of complex reflection groups and proved that their centralizers are again complex reflection groups. Using that result we show that if $\mathscr{V}$ is a finite dimensional real vector space, W is an irreducible reflection group in $\mathscr{V}$, and $\phi \in \mathrm{Aut}(\Phi_W)$ with characteristic polynomial $(x^2 - 2\cos(\frac{\pi}{d})x + 1)^n$, then $C_W(\phi)$ is a complex reflection group in V, where $\rho_{2d} = e^{\pi i/d}$ and V is the complex vector space defined via (1) ${}_{\mathbf{R}}V = \mathscr{V}$ and (2) $\rho_{2d} v = \phi(v)$, $v \in V$ (Theorem 3.2).

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On the other hand, if $G < W$, then W is irreducible (Proposition 3.1), roots of unity act as orthogonal transformations (Proposition 3.2), and there exists a $d \in \{2, 3, 4, 5\}$ such that ρ_{2d} is an automorphism of Φ_W (Corollary 3.1). Thus $C_W(\rho_{2d})$ is a complex reflection group that contains G, so the problem reduces to identifying such centralizers. To do that we rely heavily on the fact that if $G < W$ has reflections of orders d and e, $d < e$, then $d = 2$ and $e = 4$ (Theorem 3.1).

Throughout this paper, both real and complex reflections are discussed. To distinguish them, real reflections are denoted by subscripted lowercase esses: s_α, s_β, etc., while complex reflections are denoted with greek letters: σ, τ.

2. PRELIMINARIES

2.1. *Real Reflection Groups*

[2, 7] are common references for real reflection groups. Here we gather together in tabular form a summary of facts needed later.

Table 1 lists the finite irreducible real reflection groups encountered later together with their root systems and orthogonal root system automorphisms. The root systems are described with respect to an orthonormal basis.

We will need to know which conjugacy classes in $\mathrm{Aut}(\Phi_W)$ have characteristic polynomials of the form $(x^2 - 2\cos(\frac{2\pi}{d})x + 1)^n$, where $\mathrm{rank}(W) = 2n$. Table 2 lists those polynomials and the notation used by different authors to refer to the corresponding conjugacy classes. The notations $-A_2^3$ and $-A_2^4$ do not appear in [4], but they refer to the conjugacy classes with representatives $-g$ and $-h$, where g and h are representatives of A_2^3 and A_2^4, respectively.

2.2. *Complex Reflection Groups*

Let G be a complex reflection group in a complex vector space V. Then G is called *complexified* if there is a G-invariant $\mathbf{R}$-subspace V_0 of V such that the canonical map $\mathbf{C} \otimes V_0 \to V$ is bijective. We are concerned only with complex reflection groups that are truly complex, not complexified. The group G is called *imprimitive* if V is a direct sum $V = V_1 \oplus V_2 \oplus \cdots \oplus V_k$ of nontrivial proper linear subspaces V_i of V, $1 \leq i \leq k$, such that $\{V_1, V_2, \ldots, V_k\}$ is invariant under G; G is *primitive* if it is not imprimitive.

Table 3 lists the truly complex irreducible primitive reflection groups as well as some imprimitive families encountered later. The notation of the

TABLE 1
Some Irreducible Real Reflection Groups and Their Orthogonal Root System Automorphisms

W	Φ_W		$\mathrm{Aut}(\Phi_W)$
B_{2n}	$\dfrac{\pm e_i \pm e_j}{\sqrt{2}}$	$1 \le i \ne j \le 2n$	B_{2n}
	$\pm e_i$	$1 \le i \le 2n$	
D_{2n}	$\dfrac{\pm e_i \pm e_j}{\sqrt{2}}$	$1 \le i \ne j \le 2n$	B_{2n}
$n > 2$			
D_4	$\dfrac{\pm e_i \pm e_j}{\sqrt{2}}$	$1 \le i \ne j \le 4$	$D_4 \rtimes S_3 = F_4$
E_6	$\dfrac{\pm e_i \pm e_j}{\sqrt{2}}$	$1 \le i < j \le 5$	$E_6 \rtimes \mathbf{Z}_2$
	$\pm \dfrac{1}{2\sqrt{2}}(e_8 - e_7 - e_6 + \Sigma_{i=1}^5 \varepsilon_i e_i)$	$\varepsilon_i = \pm 1$, $\Pi_{i=1}^5 \varepsilon_i = -1$	
E_8	$\dfrac{\pm e_i \pm e_j}{\sqrt{2}}$	$1 \le i < j \le 8$	E_8
	$\pm \dfrac{1}{2\sqrt{2}} \Sigma_{i=1}^8 \varepsilon_i e_i$	$\varepsilon_i = \pm 1$, $\Pi_{i=1}^8 \varepsilon_i = 1$	
F_4	$\dfrac{\pm e_i \pm e_j}{\sqrt{2}}$	$1 \le i \ne j \le 4$	$F_4 \rtimes \mathbf{Z}_2$
	$\pm e_i$	$1 \le i \le 4$	
	$\frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)$		
H_4	$((\pm 1, 0, 0, 0))$*		H_4
	$((\pm b, \pm \frac{1}{2}, \pm a, 0))$	$a = \cos\left(\dfrac{\pi}{5}\right)$,	
	$(\pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2})$	$b = \cos\left(\dfrac{2\pi}{5}\right)$	

* The double parentheses indicate all even permutations of the entries.

primitive groups is from [9]. The missing groups G_{23}, G_{28}, and G_{30} are the complexified Coxeter groups H_3, F_4, and H_4, respectively.

3. CLASSIFICATION

We begin with a series of simple results which ultimately restrict both G and W: W must be irreducible (Proposition 3.1) and not isomorphic to A_{2n} (Proposition 3.7); G may contain reflections of only one order, unless it has order 4 reflections (mixing theorem). The main tool used in this

TABLE 2
Characteristic Polynomials of the Form $(x^2 - 2\cos(\frac{2\pi}{d})x + 1)^n$

W	Charpoly	Carter [4]	Grove [6]	Shinoda [10]
B_{2n}	$(x^2+1)^n$	B_2^n	—	—
D_4	$(x^2+1)^2$	$D_4(a_1)$	—	—
D_{2n}	$(x^2+1)^n$	B_2^n	—	—
E_6	$(x^2-x+1)^3$	$-A_2^3$	—	—
E_8	$(x^2+1)^4$	$D_4(a_1)^2$	—	—
	$(x^2-x+1)^4$	$-A_2^4$	—	—
F_4	$(x^2+1)^2$	$D_4(a_1)$	—	—
	$(x^2+1)^2$	—	—	w_8
	$(x^2-x+1)^2$	$F_4(a_1)$	—	—
	$(x^2-\sqrt{2}x+1)^2$	—	—	—
H_4	$(x^2+1)^2$	—	$C_1 \times C_1$	—
	$(x^2-x+1)^2$	—	$C_1 \times C_4$	—
	$(x^2-2\cos(\frac{2\pi}{10})x+1)^2$	—	$C_1 \times C_8$	—

context is the fact that roots of unity act as orthogonal transformations (Proposition 3.2). We finish by finding all the complex reflection subgroups of F_4, B_4, D_4, H_4, E_6, E_8, and finally B_{2n} and D_{2n} for $n > 2$.

3.1. *Tools*

Throughout, $\mathbf{R}$ denotes the set of real numbers and $\mathbf{C}$ the set of complex numbers. The symbol $\underline{n}$ refers to the set $\{1, 2, \dots, n\}$, and $G < W$ is an abbreviation for (V, G, W).

PROPOSITION 3.1. *If $G < W$, then W is irreducible.*

Proof. It is enough to prove that if $G < W$, then G is a complexified Coxeter group or ${}_{\mathbf{R}}V$ is $\mathbf{R}G$-irreducible, so assume ${}_{\mathbf{R}}V$ is $\mathbf{R}G$-reducible and choose a simple $\mathbf{R}G$-submodule V_0 of ${}_{\mathbf{R}}V$. Since V is $\mathbf{C}G$-irreducible, $iV_0 \oplus V_0 = V$ and $\phi\colon \mathbf{C} \otimes_{\mathbf{R}} V_0 \to iV_0 \oplus V_0$, $\lambda \oplus v \mapsto (\mathrm{Im}(\lambda)v, \mathrm{Re}(\lambda)v)$ is bijective, so G is complexified. ∎

PROPOSITION 3.2. *If $G < W$, then roots of unit act as orthogonal transformations.*

Proof. The result follows from the fact that $\mathrm{End}_{\mathbf{R}G}({}_{\mathbf{R}}V) = \mathbf{C}$. Indeed, assume $\mathrm{End}_{\mathbf{R}G}({}_{\mathbf{R}}V) = \mathbf{C}$ and let $(\ ,\ )$ be a W-invariant inner product on ${}_{\mathbf{R}}V$, and let $\langle\ ,\ \rangle$ be a G-invariant unitary inner product on V. Since $(\ ,\ )$ and $\mathrm{Re}\langle\ ,\ \rangle$ are both G-invariant and non-degenerate, there exists a G-equivariant linear map ϕ such that $(x, y) = \mathrm{Re}\langle x, \phi(y)\rangle$, and since $\mathrm{End}_{\mathbf{R}G}({}_{\mathbf{R}}V) = \mathbf{C}$, $(x, y) = \mathrm{Re}\langle x, \phi(y)\rangle = \mathrm{Re}(\langle x, \lambda y\rangle) = \mathrm{Re}(\overline{\lambda}\langle x, y\rangle)$ for

TABLE 3
Primitive Complex Reflection Groups and Some Imprimitive Families [5]

G	Rank(G)	$\lvert G\rvert$	$\lvert Z(G)\rvert$	Number of reflections of order 2	3	4	5
G_4	2	24	2	0	8	0	0
G_5	2	72	6	0	16	0	0
G_6	2	48	4	6	8	0	0
G_7	2	144	12	6	16	0	0
G_8	2	96	4	6	0	12	0
G_9	2	192	8	18	0	12	0
G_{10}	2	288	12	6	16	12	0
G_{11}	2	576	24	18	16	12	0
G_{12}	2	48	2	12	0	0	0
G_{13}	2	96	4	18	0	0	0
G_{14}	2	144	6	12	16	0	0
G_{15}	2	288	12	18	16	0	0
G_{16}	2	600	10	0	0	0	48
G_{17}	2	1200	20	30	0	0	48
G_{18}	2	1800	30	0	40	0	48
G_{19}	2	3600	60	30	40	0	48
G_{20}	2	360	6	0	40	0	0
G_{21}	2	720	12	30	40	0	0
G_{22}	2	240	4	30	0	0	0
G_{24}	3	336	2	21	0	0	0
G_{25}	3	648	3	0	24	0	0
G_{26}	3	1296	6	9	24	0	0
G_{27}	3	2160	6	45	0	0	0
G_{29}	4	7680	4	40	0	0	0
G_{31}	4	46080	4	60	0	0	0
G_{32}	4	155520	6	0	80	0	0
G_{33}	5	51,840	2	45	0	0	0
G_{34}	6	39,191,040	6	126	0	0	0
$G(4,1,n)$	n	$4^n n!$	4	$n(2n-1)$	0	$2n$	0
$G(4,2,n)$	n	$2\cdot 4^{n-1}n!$	2	$n(2n-1)$	0	0	0
$G(m,m,n)$	n	$m^{n-1}n!$	$\gcd(m,n)$	$\frac{m(n-1)n}{2}$	0	0	0

some $\lambda \in \mathbf{C}$. In particular, if ρ is a root of unity, then $(\rho x, \rho x) = \mathrm{Re}(\bar{\lambda}\langle \rho x, \rho x\rangle) = \mathrm{Re}(\bar{\lambda}\rho\bar{\rho}\langle x, x\rangle) = \mathrm{Re}(\bar{\lambda}\langle x, x\rangle) = (x, x)$.

To prove $\mathrm{End}_{\mathbf{R}G}({}_{\mathbf{R}}V) = \mathbf{C}$, put $A = \mathrm{End}_{\mathbf{R}G}({}_{\mathbf{R}}V)$. Since G and V are complex, $\mathbf{C} \subset A$, so $\dim_{\mathbf{C}}(A) \geq 1$. Let g be a reflection and let $V = V_1 \oplus V_\rho$ be the decomposition of V into g-eigenspaces. Since $V_\rho = \{v \mid g^2 v - (\rho + \bar{\rho})gv + v = 0\}$ and $\rho + \bar{\rho} \in \mathbf{R}$, $AV_\rho = V_\rho$. But the proof of Proposition 3.1 implies that A is an $\mathbf{R}$-division ring, so $\mathbf{C} \hookrightarrow A \hookrightarrow \mathrm{End}_{\mathbf{R}}(V_\rho) \simeq M_2(\mathbf{R})$.

If $\dim_{\mathbf{C}}(A) = 2$ then $A \simeq M_2(\mathbf{R})$. Since $M_2(\mathbf{R})$ is not a division ring, $\dim_{\mathbf{C}}(A) = 1$. ∎

Remark 3.1. Proposition 3.2 strongly links the reflections of G to certain rank 2 root subsystems of Φ_W: the eigenspaces of nontrivial eigenvalues contain rank 2 root subsystems of Φ_W. More specifically, let $\sigma \in G < W$ be an order d reflection and $\rho_{2d} = e^{\pi i/d}$. Choose $\alpha, \beta \in \Phi_W$ such that $\sigma = s_\alpha s_\beta$ [4, Lemma 2]. Then the angles between α and $\pm\beta$ are $\frac{\pi}{d} \cdot n$ and $\pi - \frac{\pi}{d} \cdot n$, where n is an integer relatively prime to d, so the possible rank two root subsystems containing α and β are limited. Indeed, L_σ (the σ-eigenspace corresponding to the nontrivial eigenvalue) is a complex subspace, so ${}_\alpha\Phi_\sigma := \{\alpha, \rho_{2d}\alpha, \ldots, \rho_{2d}^{(2d-1)}\alpha\} \subset L_\sigma$, and since ρ_{2d} acts orthogonally, ${}_\alpha\Phi_\sigma \subset \Phi_W$, so ${}_\alpha\Phi_\sigma$ is a rank 2 root subsystem of Φ_W. Thus, since $\dim(V) > 1$, $\dim(W) > 2$, so $d \in \{2, 3, 4, 5\}$.

When $d \neq 2$, ${}_\alpha\Phi_\sigma = \Phi_W \cap L_\sigma$ and is determined by σ:

$$
{}_\alpha\Phi_\sigma = \Phi_W \cap L_\sigma = \begin{cases} \Phi_{A_2}, & \text{if } d = 3 \\ \Phi_{B_2}, & \text{if } d = 4 \\ \Phi_{H_2}, & \text{if } d = 5. \end{cases}
$$

If $d = 2$, then ${}_\alpha\Phi_\sigma = \Phi_{A_1 \times A_1}$, but $\Phi_W \cap L_\sigma$ may be $\Phi_{A_1 \times A_1}$ of Φ_{B_2}. If $\Phi_W \cap L_\sigma = \Phi_{A_1 \times A_1}$, then ${}_\alpha\Phi_\sigma$ again depends only on σ, but if $\Phi_W \cap L_\sigma = \Phi_{B_2}$, it is possible that $s_\alpha s_\beta = \sigma = s_\gamma s_\delta$ and ${}_\alpha\Phi_\sigma$, ${}_\gamma\Phi_\sigma \subset \Phi_W \cap L_\sigma$, but ${}_\alpha\Phi_\sigma \cap {}_\gamma\Phi_\sigma = \emptyset$.

PROPOSITION 3.3. *Let $G < W$, $\rho_{2d} = e^{\pi i/d}$, $\Phi_d = \{\alpha \in \Phi_W \mid \rho_{2d}\alpha \in \Phi_W\}$, and $W_d = \langle s_\alpha \mid \alpha \in \Phi_d \rangle$. Then Φ_d is a root subsystem of Φ_w.*

Proof. Put $\overline{\Phi}_d = \{w(\beta) \mid w \in W_d, \beta \in \Phi_d\}$. Then $\overline{\Phi}_d$ is a root system for W_d that contains Φ_d. In fact $\overline{\Phi}_d = \Phi_d$. To see this, let $\alpha \in \Phi_d$. Then $\rho_{2d}\alpha$ and α span a rank 2 root subsystem, Φ^2. Since ρ_{2d} acts orthogonally, $\rho_{2d}(\rho_{2d}\alpha) \in \Phi^2$, so $\rho_{2d}\Phi_d = \Phi_d$. Then $\rho_{2d} W_d \rho_{2d}^{-1} = W_d$ which implies $\rho_{2d}\overline{\Phi}_d = \overline{\Phi}_d$, so $\overline{\Phi}_d \subset \Phi_d$. ∎

The following theorem is vital. It implies that if $G < W$, then all the reflections in G have the same order, unless G has order 4 reflections, in which case G has order 4 and order 2 reflections.

THEOREM 3.1 (mixing theorem). *If $G < W$ has reflections of orders d and e, $d < e$, then $d = 2$ and $e = 4$.*

Proof. The theorem follows from the rank 2 case. Indeed, assume the rank 2 case is true and that $\operatorname{rank}(G) > 2$. Let $G(d)$ and $G(e)$ denote the subgroups of G generated by its order d and order e reflections, respectively. Since $G(d)$ and $G(e)$ are normal, they have full rank in G, so not all the roots of reflections in $G(d)$ can be orthogonal to all those of $G(e)$. Choose reflections $\sigma \in G(d)$, $\tau \in G(e)$ corresponding to non-orthogonal roots. If $G(d)$ and $G(e)$ commute, $\sigma\tau$ is a reflection whose order must be 2, 3, 4, or 5, so $d = 2$ and $e = 4$. If σ and τ do not commute, then $\langle \sigma, \tau \rangle$ is an irreducible rank 2 reflection group, so $d = 2$ and $e = 4$ by assumption.

Assume G has rank 2. If G is imprimitive, then $G = G(m, p, 2)$, so let $\{V_1, V_2\}$ be an imprimitivity system and $\sigma = s_\gamma s_{i\gamma}$ a reflection which interchanges V_1 and V_2. Put $\Phi' = V_1 \cap \Phi_W$, $\Phi'' = V_2 \cap \Phi_W$. Then Φ' and Φ'' are (isomorphic) irreducible rank 2 root subsystems. Choose $\alpha \in \Phi'$ such that $(\alpha, \gamma) \neq 0 \neq (i\alpha, \gamma)$ and put $\beta = \sigma(\alpha)$. Then $\mathscr{B} := (\alpha, i\alpha, \beta, i\beta)$ is a real ordered basis for V. Let $x = (\gamma, \alpha)$ and $y = (\gamma, i\alpha)$. Since $\sigma(\alpha) = \beta$ and $\sigma(\gamma) = -\gamma$, $\gamma = (x, y, -x, -y)_{\mathscr{B}}$, so $x^2 + y^2 = \frac{1}{2}$. Since $\{\alpha, \gamma\}$ is contained in a root system, $x, y \in \{\pm \frac{1}{2}, \pm 1/\sqrt{2}, \pm \cos(\frac{\pi}{5}), \pm\cos(\frac{2\pi}{5})\}$. But $|(\alpha, \gamma)|^2 + |(\beta, \gamma)|^2 = \frac{1}{2}$, so $x, y \in \{\pm \frac{1}{2}\}$.

Let γ_1 be the projection of γ along V_1. Then $\|\gamma_1\| = 1/\sqrt{2}$, and for $\delta \in \Phi'$, $(\delta, \gamma) = (\alpha, \gamma_1)$. If $(\delta, \gamma) \neq 0 \neq (i\delta, \gamma)$, then $|(\delta, \gamma_1)| = \frac{1}{2}$, so $|\cos(\theta_1)| = 1/\sqrt{2}$, where θ_1 is the angle between γ_1 and δ. Otherwise one of δ or $i\delta$ is orthogonal to γ. In any case, $\frac{\pi}{4}$ is an angle between roots in Φ', so $\Phi' = \Phi_{B_2}$. Let τ be a reflection of order e. Since τ stabilizes Φ' and Φ'', $e = 4$ and $d = 2$.

Since $G(e) \triangleleft G$, if G is primitive, then $G(e)$ is irreducible for otherwise G would be imprimitive. Since $G(e)$ has reflections of order $e > 2$, $G(e)$ is not a complexified Coxeter group, so $G(e) < W_e$ implies W_e and hence Φ_e is irreducible by Proposition 3.1. If $\Phi_e = \Phi_W$, then $\rho_{2e}\alpha \in \Phi_W$ for all $\alpha \in \Phi_W$, so

$$\mathbf{C}\alpha \cap \Phi_W = \begin{cases} \Phi_{A_2}, & \text{if } e = 3 \\ \Phi_{B_2}, & \text{if } e = 4 \\ \Phi_{H_2}, & \text{if } e = 5, \end{cases}$$

so the theorem follows. If $\Phi_e \neq \Phi_W$, then $\rho_{2e} \in \mathrm{Aut}(\Phi_e)\backslash W$. Moreover, Φ_e is an irreducible rank 4 root system that is properly contained in another irreducible rank 4 root system. The possible pairs (Φ_e, Φ_W) are (Φ_{D_4}, Φ_{F_4}), (Φ_{D_4}, Φ_{B_4}), (Φ_{D_4}, Φ_{H_4}), (Φ_{B_4}, Φ_{F_4}), and (Φ_{A_4}, Φ_{H_4}). Now, (Φ_{D_4}, Φ_{F_4}), (Φ_{B_4}, Φ_{F_4}), and (Φ_{A_4}, Φ_{H_4}) are ruled out since $\mathrm{Aut}(\Phi_{D_4})$, $\mathrm{Aut}(\Phi_{B_4}) \subset F_4$, and $\mathrm{Aut}(\Phi_{A_4}) \subset H_4$. Before considering (Φ_{D_4}, Φ_{B_4}) and (Φ_{D_4}, Φ_{H_4}) individually, note that neither $\frac{\pi}{4}$ nor $\frac{\pi}{5}$ is an angle between roots of Φ_{D_4}, so if $\Phi_e = \Phi_{D_4}$, then $e = 3$ and $d = 2$.

Assume $(\Phi_3, \Phi_W) = (\Phi_{D_4}, \Phi_{B_4})$. Since Φ_2 has rank 4 and $\Phi_2 \cap \Phi_3 = \emptyset$, $\Phi_2 = \Phi_{A_1 \times A_1 \times A_1 \times A}$, so $G(2) \subset W_2 = A_1 \times A_1 \times A_1 \times A_1$ is Abelian and therefore reducible. But $G(2)$ reducible implies that G is imprimitive, a contradiction.

Assume $(\Phi_3, \Phi_W) = (\Phi_{D_4}, \Phi_{H_4})$. Let k be the order of $\rho_6 \in \mathrm{Aut}(\Phi_{D_4})/D_4 \simeq S_3$. Then $\rho_6^k \in D_4$ and has characteristic polynomial $(x^2 - 2\cos(\frac{2\pi k}{6})x + 1)^2$.

For $k = 1$ or $k = 2$, D_4 does not have an element with the required characteristic polynomial, so $k = 3$. But order 3 outer automorphisms of $\Phi_{D_4} \subset \Phi_{H_4}$ are contained in a subgroup $H_3 \leq H_4$, a contradiction. To see this let Γ be the Coxeter diagram $\Phi_{D_4} \subset \Phi_{H_4}$. Since Φ_{H_4} is a single root orbit, we may choose the middle vertex of Γ to be e_4, so an order 3 graph automorphism must fix e_4 and permute $\{\pm e_1, \pm e_2, \pm e_3\}$. Now, $\{\alpha \in \Phi_{H_4} \mid (\alpha, e_4) = 0\}$ is a root system of type H_3, and the corresponding reflection group, H_3, contains all order 3 permutations of $\{\pm e_1, \pm e_2, \pm e_3\}$. Indeed, since s_{e_1}, s_{e_2}, and s_{e_3} are in H_3, it is enough to show that $e_1 \to e_2 \to e_3 \to e_1$ is in H_3. There are four elements in H_3 which send e_1 to e_2. Since these elements fix e_4, they must permute $\{\pm e_1, \pm e_2, \pm e_3\}$, so multiplying by members of $\{s_{e_1}, s_{e_2}, s_{e_3}\}$ as necessary, there are two possibilities:

$$e_1 \to e_2 \to e_1, \qquad e_3 \to e_3 \qquad \text{and} \qquad e_1 \to e_2 \to e_3 \to e_1.$$

The first case would imply that $s_{e_1 - e_2} \in H_4$, a contradiction, so $e_1 \to e_2 \to e_3 \to e_1$ is in H_3. ∎

COROLLARY 3.1. *If $G < W$, then there exists $d \in \{2,3,4,5\}$ such that $\Phi_W = \Phi_d$. Therefore, if ρ_{2d} denotes the* **R**-*linear map $v \mapsto e^{\pi i/d}v$, then $\rho_{2d} \in \mathrm{Aut}(\Phi_W)$.*

Proof. Let $G < W$. If G has solely order d reflections, then $G < W_d$, so since W is minimal, $W = W_d$, and $\Phi_W = \Phi_d$. If G has an order 4 reflection, then $\Phi_4 \subseteq \Phi_2$, so $G < W_2$ and $\Phi_W = \Phi_2$. ∎

Remark 3.2. Springer defined *regular* elements in complex reflection groups and proved that their centralizers are again complex reflection groups [11, Theorem 4.2]. Below, the fact that $\rho_{2d} \in \mathrm{Aut}(\Phi_W)$ ($d \in \{2,3,4,5\}$, as above) is used to prove that $1 \otimes \rho_{2d}$ is a regular element in the complex reflection group $1 \otimes W$ so that $C_{1 \otimes W}(1 \otimes \rho_{2d})$ is a complex reflection group. It then follows that $C_W(\rho_{2d})$ is a complex reflection group, which reduces the problem to identifying such centralizers and determining their subgroups. ∎

LEMMA 3.1. *Let W be an irreducible rank $2n$ $(n > 1)$ real reflection group and $d \in \{2,3,4,5\}$. If $\phi \in \mathrm{Aut}(\Phi_W)$ has characteristic polynomial $(x^2 - 2\cos(\frac{\pi}{d})x + 1)^n$, then ϕ is regular.*

Proof. If $W = E_6$, E_8, F_4, or H_4, the result follows from [11, Sects. 5 and 6] via case-by-case inspection. If $W = A_{2n}$, the result follows from the proof of Proposition 3.7 below. For $W = B_{2n}$, note that its elements have characteristic polynomials of the form $\prod(x^{n_i} \pm 1)$, where $\Sigma n_i = 2n$, so $\phi \in \mathrm{Aut}(B_{2n}) = B_{2n}$ implies $d = 2$, in which case ϕ is regular [11, Sects. 5 and 6]. Finally, the case $W = D_{2n}$ follows from the above arguments since $\mathrm{Aut}(\Phi_{D_4}) = F_4$ and $\mathrm{Aut}(\Phi_{D_{2n}}) = B_{2n}$ for $n > 2$. ∎

THEOREM 3.2. *Let $\mathscr{V}$ be a $2n$-dimensional real vector space $(n > 1)$, let W be an irreducible rank $2n$ reflection group in $\mathscr{V}$, and let $\phi \in \mathrm{Aut}(\Phi_W)$ with characteristic polynomial $(x^2 - 2\cos(\frac{\pi}{d})x + 1)^n$. Define a complex vector space V via (1) ${}_{\mathbf{R}}V = \mathscr{V}$ and (2) $\rho_{2d}v = \phi(v)$, where $\rho_{2d} = e^{\pi i/d}$. Then $C_W(\phi)$ is a complex reflection group in V.*

Proof. Apply Springer's theorem [11, Theorem 4.2] to the complex reflection group $1 \otimes W$ in $U := \mathbf{C} \otimes_{\mathbf{R}} V$. By the above lemma, $1 \otimes \phi$ is regular, so $C_{1\otimes W}(1 \otimes \phi)$ is a complex reflection group in U_λ, where $U = U_\lambda \otimes U_{\bar{\lambda}}$ is the decomposition of U into $1 \otimes \phi$ eigenspaces [11, Theorem 4.2]. If $1 \otimes \sigma \in C_{1\otimes W}(1 \otimes \phi)$ is a complex reflection in U_λ, then $\mathrm{charpoly}_{{}_{\mathbf{R}}V}(\sigma) = \mathrm{charpoly}_U(1 \otimes \sigma) = \mathrm{charpoly}_{U_{\bar{\lambda}}}(1 \otimes \sigma)\mathrm{charpoly}_{U_{\bar{\lambda}}}(1 \otimes \sigma) = (x-1)^{n-1}(x-\rho)(x-1)^{n-1}(x-\bar{\rho})$, where $2n = \dim_{\mathbf{R}} V$ and ρ is the non-trivial eigenvalue of $1 \otimes \sigma$. Thus, σ fixes a complex hyperplane, and so is a complex reflection. ∎

The next four propositions are more specific than some previous results but help streamline the procedure for finding complex reflection subgroups. The first two concern the case when G is imprimitive, the third restricts W when G is primitive and has order four reflections, and the fourth states that A_{2n} has no complex reflection subgroups of rank n.

PROPOSITION 3.4. *If $G(m, p, n) < W$, then W has an element with characteristic polynomial (over $\mathbf{R}$) equal to $(x^2 - 2\cos(\frac{2\pi}{m})x + 1)^2(x-1)^{2n-4}$, and if $m \neq p$, then $m \in \{4, 8\}$.*

Proof. The group $G := G(m, p, n)$ is subgroup of both $GL_n(\mathbf{C})$ and W, so if $\sigma \in G$ is an order d reflection, it may be written as a diagonal matrix, $\sigma = d(a_1, a_2, \ldots, a_n)$, where one a_i is a primitive dth root of unity and the others are 1, but since $\sigma \in W$, it is also a product of real reflections in W, $\sigma = s_\alpha s_\beta$ [4, Lemma 2], where $\alpha, \beta \in \Phi_W$ and $(\alpha, \beta) = \pm\cos(\frac{\pi}{d})$.

Let $\{V_i \mid i \in \underline{n}]$ be an imprimitivity system for G, and let ρ_m be a primitive mth root of unity. Then $\tau := d(\rho_m, \rho_m^{-1}, 1, \ldots, 1) \in G$, and $\mathrm{charpoly}_{\mathbf{R}}(\tau) = (x^2 - 2\cos(\frac{2\pi}{m})x + 1)^2(x-1)^{2n-4}$, which proves the first statement. For the second, assume $m \neq p$ and note that since G has order 2 reflections, the mixing theorem implies $\Phi_W = \Phi_2$. But G also has reflections of order $\frac{m}{p}$, so $\frac{m}{p} = 2$ if G has only order 2 reflections, while $\frac{m}{p} = 4$ if G has order 4 reflections. In any case, m is even, so $\sigma := d(-1, 1, \dots, 1) \in G$. Choose $\alpha, \beta \in \Phi_W$ such that $\sigma = s_\alpha s_\beta$. Then $(\alpha, \beta) = 0$, so $\Phi_W \cap V_1 = \Phi_{A_1 \times A_1}$ or $\Phi_W \cap V_1 = \Phi_{B_2}$.

Since $\tau \in W$, $\tau(\Phi_W) = \Phi_W$, and $\tau(V_1) = \rho_m V_1 = V_1$, so $\tau(\Phi_W \cap V_1) = \rho_m(\Phi_W \cap V_1) = \Phi_W \cap V_1$. Therefore, since ρ_m acts orthogonally, $m \in \{2, 4, 8\}$. But $G(2,1,n)$ and $G(2,2,n)$ are complexified Coxeter groups, so $m \neq 2$. Thus, $m \in \{4, 8\}$. ∎

Proposition 3.5 is used below to limit the possible complex subgroups of H_4, E_6, and E_8.

PROPOSITION 3.5. *Let $G = G(4,2,n)$ and assume W is a real reflection group whose Coxeter diagram has no double bond. If $G < W$, then $W = D_{2n}$.*

Proof. Remark 3.1 describes how roots are associated with complex reflections. The set of all such roots corresponding to the reflections in G forms the root system $\Phi_{D_{2n}}$, so the proposition follows from the minimality of W.

Indeed, let $\{V_k \mid k \in \underline{n}]$ be an imprimitivity system for G. There are two types of reflections in G—those that act diagonally on $\{V_k \mid k \in \underline{n}\}$ and those that interchange a pair of subspaces in $\{V_k \mid k \in \underline{n}\}$. But since the Coxeter diagram of W has no double bond and all the reflections in G have order 2, the root subsystems associated with them are type $A_1 \times A_1$. To identify these roots choose an **R**-basis from the root system as follows: For each $k \in \underline{n}$, put $\Phi^k = V_k \cap \Phi_W$, and choose $\alpha_k \in \Phi^k$; then $\Phi^k = \{\pm\alpha_k, \pm i\alpha_k\}$. Fix $i, j \in \underline{n}$ and let $\sigma \in G$ be a reflection that interchanges V_i and V_j. Choose $\gamma \in \Phi_W$ such that $\sigma = s_\gamma s_{i\gamma}$, and let $(a_1, a_2, \dots, a_{2n})$ be the coordinates of γ with respect to the ordered **R**-basis $(\alpha_1, i\alpha_1, \alpha_2, i\alpha_2, \dots, \alpha_n, i\alpha_n)$. The coordinates of γ are inner products of roots, for example, $a_1 = (\gamma, \alpha)$, and the Coxeter diagram of W has no double bond, so the a_i are in $\{0, \pm\frac{1}{2}\}$. Since σ fixes V_k, $k \neq i, j$, there are at most four non-zero coordinates for γ, which must be $\pm\frac{1}{2}$, so there are at most 16 such roots. In fact there are exactly 16, since there are four reflections that interchange V_i and V_j, each corresponding to four roots. Thus, the coordinates of the roots corresponding to reflections that interchange a pair of subspaces in $\{V_k \mid k \in \underline{n}\}$ have exactly four nonzero entries taken from $\{\pm\frac{1}{2}\}$. For example, the coordinates for the roots corresponding to the reflections that interchange V_1 and V_2 are $(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, 0, \dots, 0)$. The coordinates of the roots corresponding to the diagonal reflections are $(-1, -1, 0, \dots, 0), (0, 0, -1, -1, 0, \dots, 0), \dots, (0, \dots, 0, -1, -1)$.

Define a new ordered **R**-basis $\{e_i : i \in \underline{2n}\}$ via $e_{2k-1} = (\alpha_k + i\alpha_k)/2$ and $e_{2k} = (\alpha_k - i\alpha_k)/2$, for $k \in \underline{n}$. With respect to this basis the roots corresponding to the reflections in G are $\{\pm e_i \pm e_j : 1 \leq i, j \leq 2n\}$. ∎

Proposition 3.6 restricts possible primitive subgroups for it implies that if $G < W$ is primitive, then both ρ_4 and ρ_8 are in $\mathrm{Aut}(\Phi_W)$.

PROPOSITION 3.6. *If $G < W$ is primitive and has order 4 reflections, then* $\Phi_4 = \Phi_2 = \Phi_W$.

Proof. The last equality follows from the minimality of W. To prove the first, note that no primitive group of rank greater than 2 has an order 4 reflection, so assume G has rank 2. The mixing theorem implies that G has only order 2 and order 4 reflections; there are only two such groups: G_8 and G_9 (Table 3). Since all its order 2 reflections are squares of order 4 reflections, G_8 satisfies the theorem. Now, G_9 has 18 order 2 reflections, and each corresponds to a root subsystem of type $A_1 \times A_1$ or B_2, so Φ_W must have at least $18 \cdot 4 = 72$ roots. But the largest rank 4 root system whose Coxeter diagram has a double bond is Φ_{F_4}, which has only 48 roots. ∎

PROPOSITION 3.7. *A_{2n} has no complex reflection subgroups of rank n.*

Proof. If $G < A_{2n}$, then there exists a $d \in \{2,3,4,5\}$ such that $\Phi_d = \Phi_{A_{2n}}$, so $\rho_{2d} \in \mathrm{Aut}(\Phi_{A_{2n}}) = A_{2n} \rtimes \{\pm 1\}$. Elements of A_{2n} have characteristic polynomials of the form $(x^{\lambda_1} - 1)(x^{\lambda_2} - 1) \cdots (x^{\lambda_k} - 1)/(x - 1)$, $\lambda_1 + \lambda_2 + \cdots + \lambda_k = 2n + 1$, and $\mathrm{charpoly}_{\mathbf{R}}(\rho_{2d}) = (x^2 - 2\cos(\frac{\pi}{d})x + 1)^n$. But $(x - 1)$ divides all the $x^{\lambda_i} - 1$, while 1 is not a root of $x^2 - 2\cos(\frac{\pi}{d})x + 1$, so $k = 1$; thus $\rho_{2d} \notin A_{2n}$. The elements in $\mathrm{Aut}(\Phi_{A_{2n}}) \setminus A_{2n}$ can be expressed as $-1_V \cdot \sigma$, where $-1_V\colon V \to V$, $v \mapsto -v$, and $\sigma \in A_{2n}$. Since $2n$ is even, $\mathrm{charpoly}_{\mathbf{R}}(-1_V \cdot \sigma) = p(-x)$, where $p(x) = \mathrm{charpoly}_{\mathbf{R}}(\sigma)$, so the above argument applies again; thus $\rho_{2d} \notin \mathrm{Aut}(\Phi_{A_{2n}})$. ∎

3.2. *Classification*

Now we describe a general method for finding complex and real reflection groups G and W that satisfy $G < W$.

Remark 3.1 implies that the possible orders of reflections in any complex reflection subgroup of W are determined by the rank 2 root subsystems of Φ_W:

$$\Phi_{A_1 \times A_1} \leftrightarrow \text{order } 2$$
$$\Phi_{A_2} \leftrightarrow \text{order } 3$$
$$\Phi_{H_2} \leftrightarrow \text{order } 5$$
$$\Phi_{B_2} \leftrightarrow \text{order 2 or 4.}$$

The mixing theorem states that if $G < W$, then all the reflections in G have the same order or that G has reflections of orders 2 and 4. Moreover, Corollary 3.1 says that if G has reflections of a single order d, then $\Phi_W = \Phi_d$, while $\Phi_W = \Phi_2$ otherwise, so if $G < W$, there exists $d \in \{2,3,4,5\}$ such that $\rho_{2d} \in \operatorname{Aut}(\Phi_W)$ and $G < C_W(\rho_{2d})$. Since Theorem 3.2 states that centralizers of the form $C_W(\rho_{2d})$ are complex reflection groups, to find the complex reflection subgroups of W it is enough to identify such centralizers and their reflection subgroups.

To identify a centralizer $C_W(\rho_{2d})$ we will usually try to decide: (1) whether $C_W(\rho_{2d})$ is primitive or imprimitive, and (2) how many order d reflections it contains. For (1), note that if $\Phi_W = \Phi_d$, $d \neq 2$, then $C_W(\rho_{2d})$ contains only order d reflections so must be primitive, since imprimitive groups have order 2 reflections. If $\Phi_W = \Phi_2$, then $C_W(\rho_{2d})$ has order 2 reflections but may or may not have order 4 reflections and may or may not be primitive. For (2), recall that an order d reflection $\sigma = s_\alpha s_\beta$ corresponds to a rank 2 root subsystem ${}_\alpha\Phi_\sigma$ of Φ_W. In fact $\sigma, \sigma^2, \dots, \sigma^{d-1}$ all correspond to ${}_\alpha\Phi_\sigma$. Moreover, any two order d reflections correspond to the same rank 2 root subsystem or to disjoint subsystems since the subsystems are contained in the eigenspaces for the nontrivial eigenvalues of the respective reflections. Therefore, $(|\Phi_W|/|{}_\alpha\Phi_\sigma|) \cdot \phi(d)$ is an upper bound for the number of order d reflections in $C_W(\rho_{2d})$, where ϕ is the Euler phi function. If the Coxeter diagram of W has no double bond, then $C_W(\rho_{2d})$ has exactly $(|\Phi_W|/|{}_\alpha\Phi_\sigma|) \cdot \phi(d)$ order d reflections since $\Phi_W = \Phi_d$. This is still true if the Coxeter diagram has a double bond, so long as d is not even. But if d is even and the Coxeter diagram has a double bond, then ad hoc methods are used.

Finally, the triple $(V, C_W(\rho_{2d}), W)$ is unique up to isomorphism when there is a unique conjugacy class in $\operatorname{Aut}(\Phi_W)$ whose elements have characteristic polynomial $(x^2 - 2\cos(\frac{\pi}{d})x + 1)^n$, since $\phi \in \operatorname{Aut}(\Phi_W)$ implies $\phi W \phi^{-1} = W$ and $\phi C_W(\rho_{2d})\phi^{-1} = C_W(\phi\rho_{2d}\phi^{-1})$.

We first find the complex reflection subgroups of the rank 4, 6, and 8 real reflection groups individually and then treat the families B_{2n} and D_{2n} $(n > 2)$ uniformly. When the context justifies it, we will use Φ rather than Φ_W to denote the root system.

F_4

The rank 2 root subsystems of Φ are $A_1 \times A_1$, A_2, and B_2, so the possible orders of reflections in any complex reflection subgroup of F_4 are 2, 3, and 4, so $\Phi = \Phi_2$ or $\Phi = \Phi_3$.

- $\Phi = \Phi_3$. There is a unique conjugacy class in $\operatorname{Aut}(\Phi)$ whose elements have characteristic polynomial $(x^2 - 2\cos(\frac{2\pi}{6})x + 1)^2 = (x^2 - x +$

$1)^2$ (Table 2), so the complex reflection group $C_{F_4}(\rho_6)$ is unique up to isomorphism. Since $C_{F_4}(\rho_6)$ has $(|\Phi|/|\Phi_{A_2}|)\cdot\phi(3) = \frac{48}{6}\cdot 2 = 16$ order 3 reflections and none of other orders, $C_{F_4}(\rho_6)$ is an irreducible primitive rank 2 complex reflection group whose 16 reflections all have order 3. Therefore, $C_{F_4}(\rho_6) \simeq G_5$ (Table 3). The only reflection subgroup of G_5 is G_4, but $G_4 \subset D_4$, so F_4 is not minimal for G_4. Indeed, let $\sigma_i = s_{\alpha_i}s_{\beta_i}$ $(i = 1, 2, \ldots, 4)$ be the four order 3 reflections in G_4 with determinant ρ_3. Then $\Phi' := \bigcup_{i=1}^{4} {}_{\alpha_i}\Phi_{\sigma_i}$ contains 24 roots. Since the 3-Sylow subgroups of G_4 have order 3, the reflections of a given determinant are conjugate, so the elements in Φ' are all in the same F_4 root orbit. Therefore, Φ' is a root system of type D_4, and $G_4 \subset W(\Phi') = D_4$.

- $\Phi = \Phi_2$. There are two conjugacy classes in $\mathrm{Aut}(\Phi)$ with characteristic polynomial $(x^2 - 2\cos(\frac{2\pi}{4})x + 1)^2$; one contains inner automorphisms, the other outer. The corresponding actions of ρ_4 are denoted by ρ_4^i and ρ_4^{o}, respectively, when it is necessary to distinguish them.

The action of ρ_4 partitions Φ into 12 $A_1 \times A_1$ root subsystems of the form $\{\alpha, \rho_4\alpha, \rho_4^2\alpha, \rho_4^3\alpha\}$, so $C_{F_4}(\rho_4)$ has at most 12 order 2 reflections. There are less than 12 exactly when pairs of those subsystems together form B_2 subsystems. Since orthogonal roots in a B_2 root subsystem of F_4 are in the same F_4 orbit, $C(\rho_4^{\mathrm{o}})$ has 12 order two reflections and $C(\rho_4^i)$ has 6. Thus, $C(\rho_4^{\mathrm{o}}) \simeq G_{12}$ (Table 3). To identify $C(\rho_4^i)$ note that the action of ρ_8 is given by an element from the conjugacy class in $\mathrm{Aut}(\Phi)$ with characteristic polynomial $(x^2 - 2\cos(\frac{2\pi}{8})x + 1)^2$, and since $\mathrm{Aut}(\Phi) = F_4 \rtimes \mathbf{Z}_2$, we may and shall assume that $\rho_8^2 = \rho_4^i$. Since $C_W(\rho_8)$ has $\frac{48}{8}\cdot 2 = 12$ order 4 reflections, it is isomorphic to G_8 (Table 3), and $C_W(\rho_8) \subset C_W(\rho_8^2) = C_W(\rho_4^i)$, so $C_W(\rho_4^i) = C_W(\rho_8) \simeq G_8$.

Neither G_{12} nor G_8 have primitive reflection subgroups (Table 3), and since F_4 does not have an element with characteristic polynomial $(x^2 - 2\cos(\frac{2\pi}{8})x + 1)^2$, Proposition 3.4 implies that the only possible imprimitive subgroups of G_8 or G_{12} are $G(4,1,2)$ and $G(4,2,2)$. Since $\left(\begin{smallmatrix} i & 0 \\ 0 & i \end{smallmatrix}\right) \in G(4,2,2) \leq G(4,1,2)$ and $|Z(G_{12})| = 2$, G_{12} has no imprimitive reflection subgroups. On the other hand, the 2-Sylow subgroup of G_8 is isomorphic to $G(4,1,2)$, but F_4 is not minimal for $G(4,1,2)$. To see this, let H be the subgroup of G_8 generated by its order 2 reflections. Then H is an irreducible rank 2 complex reflection group whose 6 reflections all have order 2, so H is imprimitive (Table 3), and since H has only order 2 reflections $H = G(4,2,2)$. Now H is a 2-group, so it is contained in a 2-Sylow subgroup P_2 of G_8, and since G_8 has 12 order four reflections and three 2-Sylow subgroups, each 2-Sylow subgroup must contain at least 4 order four reflections, so $P_2 \simeq G(4,1,2)$. Let $\sigma_i = s_{\alpha_i}s_{\beta_i}$, $i = 1, 2, \ldots, 10$, be the reflections in $G(4,1,2)$, where $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ have order 4 and σ_5, σ_6 are squares of order 4 reflections. Then $\sigma_7, \sigma_8, \sigma_9, \sigma_{10}$ are conjugate in $G(4,1,2)$, and since $G(4,1,2) < C_{F_4}(\rho_4) \simeq G_8$, where ρ_4 acts via an inner automorphism of Φ, α_i and β_i are in the same F_4 root orbit for $i \geq 7$, so $s_{\alpha_i} s_{\beta_i} = s_{(\alpha_i + \beta_i)/\sqrt{2}} s_{(\alpha_i - \beta_i)/\sqrt{2}}$. The same is true for σ_5 and σ_6, so the roots $\alpha_5, \beta_5, \ldots, \alpha_{10}, \beta_{10}$ can be chosen from the same F_4 root orbit, and therefore, the roots α_i, β_i, $i = 1, 2, \ldots, 10$, can be chosen so that $\bigcup_{i=1}^{10} {}_{\sigma_i}\Phi_{\beta_i}$ is a root system of type B_4 or C_4, so F_4 is not minimal for $G(4,1,2)$.

Remark 3.3. In the last paragraph we saw that $G(4,1,2) \subset B_4 \cap C_4$ since $\sigma_i = s_{\alpha_i} s_{\beta_i} = s_{(\alpha_i + \beta_i)/\sqrt{2}} s_{(\alpha_i - \beta_i)/\sqrt{2}}$ for $i \geq 5$. In particular, both B_4 and C_4 are minimal for $G(4,1,2) < F_4$. Something similar occurs for $G(4,2,2) < F_4$.

Let $\tau_1, \ldots, \tau_6$ be the reflections in $G(4,2,2)$ and $\{\tau_1, \tau_2\}, \{\tau_3, \tau_4, \tau_5, \tau_6\}$ the conjugacy classes. Choose $\alpha_1, \beta_1, \ldots, \alpha_6, \beta_6$ such that $\tau_i = s_{\alpha_i} s_{\beta_i}$. Since $\{\tau_3, \tau_4, \tau_5, \tau_6\}$ are conjugate, we may assume that $\{\alpha_3, \beta_3, \ldots, \alpha_6, \beta_6\}$ are in the same F_4 root orbit; similarly, $\{\alpha_1, \beta_1, \alpha_2, \beta_2\}$ are in the same orbit. Since each F_4 root orbit is a D_4 root subsystem, $s_{\alpha_i} s_{\beta_i} = s_{(\alpha_i + \beta_i)/\sqrt{2}} s_{(\alpha_i - \beta_i)/\sqrt{2}}$, for $i \geq 1$. Thus, we may choose the α_i and β_i so that $\bigcup_{i=1}^{6} {}_{\alpha_i}\Phi_{\tau_i}$ is either of the F_4 root orbits; so if D_4 and D_4' are the groups corresponding to the orbits, then $G(4,2,2) \subset D_4 \cap D_4'$.

B_4

Let $G < B_4$. Possible orders of reflections in G are 2, 3, and 4, so $\Phi = \Phi_2$ or $\Phi = \Phi_3$.

• $\Phi = \Phi_3$. Here, $\rho_6 \in \mathrm{Aut}(\Phi)$, but B_4 does not have an element with characteristic polynomial $(x^2 - 2\cos(\frac{2\pi}{6})x + 1)^2$, so there are no complex reflection subgroups in this case.

• $\Phi = \Phi_2$. There is exactly one conjugacy class in B_4 with characteristic polynomial $(x^2 - 2\cos(\frac{2\pi}{4})x + 1)^2$, so the complex reflection group $C_{B_4}(\rho_4)$ is unique. Since $D_4 \triangleleft B_4$ and has only one conjugacy class with characteristic polynomial $(x^2 - 2\cos(\frac{2\pi}{4})x + 1)^2$, $\rho_4 \in D_4$.

Suppose $C_{B_4}(\rho_4)$ is primitive. If $C_{B_4}(\rho_4)$ has an order 4 reflection, then Proposition 3.6 implies $\Phi = \Phi_2 = \Phi_4$, so $\rho_8 \in \mathrm{Aut}(\Phi)$. But B_4 does not have an element with characteristic polynomial $(x^2 - 2\cos(\frac{2\pi}{8})x + 1)^2$. On the other hand, if $C_{B_4}(\rho_4)$ has solely order 2 reflections, then $C_{B_4}(\rho_4)$ has at most eight reflections since there are only $\frac{32}{4} = 8$ ρ_4-orbits on Φ, but there is no such primitive group (Table 3).

Therefore, $C_{B_4}(\rho_4)$ is imprimitive, and since B_4 does not have an element with characteristic polynomial $(x^2 - 2\cos(\frac{2\pi}{8})x + 1)^2$, $C_{B_4}(\rho_4) = G(4,1,2)$ or $C_{B_4}(\rho_4) = G(4,2,2)$. Direct computation shows that $|C_{B_4}(\rho_4)| = 32$, so $C_{B_4}(\rho_4) = G(4,1,2)$. Now, $G(4,2,2) < G(4,1,2)$, but $G(4,2,2) < D_4$. To see this let $\sigma_i = s_{\alpha_i} s_{\beta_i}$ be the six order 2 reflections in $G(4,2,2)$ divided into conjugacy classes $\{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$ and $\{\sigma_5, \sigma_6\}$. Since the corresponding reflections are conjugate, the 16 roots in $\bigcup_{i=1}^{4} {}_{\alpha_i}\Phi_{\sigma_i}$ can be chosen from a single (long) B_4-orbit. Assume $i > 4$. If α_i and β_i are short, then $\sigma_i = s_{\alpha_i} s_{\beta_i} = s_{(\alpha_i+\beta_i)/\sqrt{2}} s_{(\alpha_i-\beta_i)/\sqrt{2}}$. Therefore, the α_i and β_i can be chosen so that the 24 roots of $\bigcup_{i=1}^{6} {}_{\alpha_i}\Phi_{\sigma_i}$ are contained in a single B_4-orbit and hence form a D_4 root subsystem.

D_4

Possible orders of reflections in $G < D_4$ are 2 and 3, so $\Phi = \Phi_2$ or $\Phi = \Phi_3$.

- $\Phi = \Phi_3$. There is only one conjugacy class in $\mathrm{Aut}(\Phi) = F_4$ with characteristic polynomial $(x^2 - 2\cos(\frac{2\pi}{6})x + 1)^3$, so the complex reflection group $C_{D_4}(\rho_6)$ is unique. Since $C_{D_4}(\rho_6)$ has $\frac{24}{6} \cdot 2 = 8$ order 3 reflections, $C_{D_4}(\rho_6) \simeq G_4$ and there are no irreducible reflection subgroups (Table 3).
- $\Phi = \Phi_2$. The complex reflection group $C_{D_4}(\rho_4)$ is unique since there is only one conjugacy class in F_4 with characteristic polynomial $(x^2 - 2\cos(\frac{2\pi}{4})x + 1)^2$. Since there are $\frac{24}{4} = 6$ ρ_4-orbits on Φ, there are at most six order 2 reflections, so $C_{D_4}(\rho_4)$ is not primitive (Table 3). Therefore, $C_{D_4}(\rho_4)$ is imprimitive, and since there can be no order 4 reflections, $C_{D_4}(\rho_4) = G(4,2,2)$. The only reflection subgroup of $G(4,2,2)$ is $G(4,4,2)$, and it is unique since it is the subgroup of $G(4,2,2)$ generated by the non-diagonal reflections.

H_4

Let $G < H_4$. Possible orders of reflections in G are 2, 3, and 5, so $\Phi = \Phi_2$, $\Phi = \Phi_3$, or $\Phi = \Phi_5$. Moreover, $\mathrm{Aut}(\Phi_{H_4}) = H_4$ [1, Appendix] and each conjugacy class in H_4 has a distinct characteristic polynomial, so the complex reflection groups $C_{H_4}(\rho_{2d})$, $d \in \{2,3,5\}$ are unique.

- $\Phi = \Phi_5$. Since $C_{H_4}(\rho_{10})$ has $\frac{120}{10} \cdot 4 = 48$ order 5 reflections, it is isomorphic to G_{16} and there are no irreducible reflection subgroups (Table 3).
- $\Phi = \Phi_3$. Since $C_{H_4}(\rho_6)$ has $\frac{120}{6} \cdot 2 = 40$ order 3 reflections, it is isomorphic to G_{20}; the only possible subgroups are G_4 and G_5. To find

them note that $Z := Z(G_{20})$ has order 6 and that $G_{20}/A \simeq \mathscr{A}_5$ (the simple group of order 60) [5]. Moreover, π: $G_{20} \to \mathscr{A}_5$ induces a bijective correspondence between the 3-Sylow subgroups of G_{20} and $\mathscr{A}_5$. Choose P_1 and P_2, 3-Sylow subgroups of $\mathscr{A}_5$, so that $\pi^{-1}(\langle P_1, P_2\rangle) \simeq G_5$. Since $G_4 < G_5$, existence of both G_4 and G_5 in $C_{H_4}(\rho_6)$ is proven; next consider uniqueness.

Let μ_3 be an order 3 element of Z. Then $G_5 = G_4 \times \langle \mu_3 \rangle$, so G_5 and G_4 have the same 2-Sylow subgroups; a counting argument shows that G_4 has only one. Let P_2 be the 2-Sylow subgroup contained in $G_4 \cap G_5$. Since $[G_{20} : G_5] = 5$, $N_{G_{20}}(P_2) = G_5$ or $N_{G_{20}}(P_2) = G_{20}$, but $P_2 \triangleleft G_{20}$ would imply that G_{20}/P_2 and hence G_{20} is solvable, which is not possible since G_{20} has a factor isomorphic to $\mathscr{A}_5$. Therefore, $G_5 = N_{G_{20}}(P_2)$, so there is a unique isomorphism class of G_5 in G_{20}. More about G_4 follows in Remark 3.4 below; for now note that $G_4 < D_4 < H_4$.

- $\Phi = \Phi_2$. Since the Coxeter diagram of H_4 has no double bond, $C_{H_4}(\rho_4)$ has $\frac{120}{4} = 30$ order 2 reflections and no order 4 reflections, so it is isomorphic to G_{22}. The only other irreducible, rank 2, primitive, complex reflection groups generated by order 2 reflections are G_{12} and G_{13}, but $|G_{13}|$ does not divide $|G_{22}|$, so consider G_{12}. Suppose $G_{12} < G_{22}$. Then they share at least one 2-Sylow subgroup, P_2. Since G_{22} is irreducible, its center, Z, contains only scalars. But Z has order 4, so $Z < P_2$, which contradicts the fact that G_{12} has a center of order 2. Thus, G_{22} has no primitive subgroups. In fact G_{22} has no imprimitive subgroups for which H_4 is minimal.

As above, let P_2 be a 2-Sylow subgroup of G_{22}. A counting argument (using the fact that $G_{22}/Z(G_{22}) \simeq \mathscr{A}_5$) shows that there are at most five 2-Sylow subgroups of G_{22}, so each must contain at least $\frac{30}{5} = 6$ order 2 reflections. Since the subgroup of P_2 generated by its reflections must be imprimitive (Table 3), it follows that $P_2 \simeq G(4, 2, 2)$, so by Proposition 3.5, it is contained in D_4.

Remark 3.4. (1) The reflections in G_5 account for 48 roots. Since $|\Phi_{F_4}| = 48$, it is not surprising that $G_5 < F_4$, but $|\Phi_{H_4}| = 120$, and Φ_{H_4} has no root subsystem containing 48 roots. Something similar happens with $G(4, 4, n)$ in D_{2n} and G_{29} in E_8.

(2) Assume $G_4 < H_4$ and let P_2 be the unique 2-Sylow subgroup of G_4. Since $G_4 < N_{G_{20}}(P_2) = G_5 < G_{20}$, it is enough to consider G_{20}-conjugacy classes of G_4 in G_5. From the discussion of G_4 in F_4, there are at least two distinct copies of G_4 in G_5. To see that there are at most two, let G_4 and G_4' be distinct subgroups of G_5; then $G_4 \cap G_4' = P_2$, so they share

no reflections. Since G_4 has 8 reflections and there are only 16 in G_5, there can be at most two copies of G_4 in G_5. Those two copies of G_4 are not conjugate in G_{20}.

Let $G_4, G_4' < G_5$ with $G_4 \neq G_4'$. Choose a reflection $\sigma \in G_4 \setminus G_4'$. Since $G_4 \times \langle \mu_3 \rangle = G_4 = G_4' \times \langle \mu_3 \rangle$, there exist $g' \in G_4'$ and $\mu_3' \in \langle \mu_3 \rangle$ such that $\sigma = g'\mu_3'$. Since g' has order 3, it is a reflection with $\det(g') \neq \det(\sigma)$. Suppose G_4 and G_4' are conjugate by an element of G_{20}. Since reflections in G_4' of a given determinant are conjugate (in G_4) there exists $x \in G_{20}$ such that $x\sigma x^{-1} = (g')^2$. Multiplying x by a suitable element from $\langle \mu_3 \rangle$, if necessary, $\det(x) = 1$. Choose a basis so that σ is represented as

$$\begin{pmatrix} \rho_3 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then corresponding to $(g')^2$ and x are

$$\begin{pmatrix} 1 & 0 \\ 0 & \rho_3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & c \\ -c^{-1} & 0 \end{pmatrix}.$$

In particular, x has order 4. Since $xG_4x^{-1} = G_4'$, $xP_2x^{-1} = P_2$, so $x \in N_{G_{20}}(P_2) = G_5$. But $G_5 = G_4 \times \langle \mu_3 \rangle$, so $x \in G_4$, a contradiction.

E_6

Possible orders of reflections in $G < E_6$ are 2 or 3, so $\Phi = \Phi_2$ or $\Phi = \Phi_3$.

- $\Phi = \Phi_3$. There is only one conjugacy class in $\mathrm{Aut}(\Phi)$ with characteristic polynomial $(x^2 - 2\cos(\frac{2\pi}{6})x + 1)^2$, so the complex reflection group $C_{E_6}(\rho_6)$ is unique. Since $C_{E_6}(\rho_6)$ has $\frac{72}{6} \cdot 2 = 24$ order 3 reflections, $C_{E_6}(\rho_6) \simeq G_{25}$ and there are no irreducible reflection subgroups (Table 3).
- $\Phi = \Phi_2$. There are no complex subgroups in this case because $\mathrm{Aut}(\Phi)$ does not have an element with characteristic polynomial $(x^2 - 2\cos(\frac{2\pi}{4})x + 1)^3$.

E_8

Possible orders of reflections in $G < E_8$ are 2 and 3, so $\Phi = \Phi_2$ or $\Phi = \Phi_3$. Moreover, $\mathrm{Aut}(\Phi) = E_8$ and there is only one conjugacy class in E_8 corresponding to the characteristic polynomials $(x^2 - 2\cos(\frac{2\pi}{4})x + 1)^4$ and $(x^2 - 2\cos(\frac{2\pi}{6})x + 1)^4$, so the complex reflection groups $C_{E_8}(\rho_4)$ and $C_{E_8}(\rho_6)$ are unique.

• $\Phi = \Phi_3$. The complex reflection group $C_{E_8}(\rho_6)$ has $\frac{240}{6} \cdot 2 = 80$ order 3 reflections, so it is isomorphic to G_{32}, and it has no reflection subgroups (Table 3).

• $\Phi = \Phi_2$. Since $C_{E_8}(\rho_4)$ has $\frac{240}{4} = 60$ order 2 reflections, it is isomorphic to G_{31}, and G_{29} is its only primitive subgroup [5, p. 409]. There are at most two isomorphism classes of G_{29} in G_{31}. To see this choose generators $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ for G_{29} corresponding to the presentation given in [3]. Choose $\alpha_i, \beta_i \in \Phi$ such that $\sigma_i = s_{\alpha_i} s_{\beta_i}$, and construct a graph Γ with vertices in one-to-one correspondence with $\{\alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3, \alpha_4, \beta_4\}$ by joining two vertices with an edge if the corresponding roots are not orthogonal; label the edge with the inner product of the roots [4]. If the inner product is $\pm \frac{1}{2}$, the edge label is often omitted. Note that since σ_i has order 2, $(\alpha_i, \beta_i) = 0$. Since the Coxeter diagram of E_8 has no double bond, the resulting graph is unique and is displayed in Fig. 1.

The numbered nodes correspond to a D_6 root subsystem and there is only one up to conjugacy in E_8, so let $e_6 - e_5$, $e_5 - e_4$, $e_4 - e_3$, $e_3 - e_2$, $e_2 - e_1$, $e_1 + e_2$ correspond to nodes 1, 2, 3, 4, 5, and 6, respectively. Then the constraints of the diagram require that up to sign change there are only four roots that can correspond to node a: $-e_4 \pm e_7$ and $-e_4 \pm e_8$. Similarly, node b may correspond to $\frac{1}{2}(-e_1 - e_2 + e_3 + e_4 - e_5 - e_6 + \omega e_7 + \varepsilon e_8)$, where $\varepsilon = \omega = \pm 1$. Since the roots corresponding to nodes a and b are orthogonal, there are four possibilities: $\{(-e_4 + e_7, -e_1 - e_2 + e_3 + e_4 - e_5 - e_6 + e_7 + e_8), (-e_4 + e_8, -e_1 - e_2 + e_3 + e_4 - e_5 - e_6 + e_7 + e_8), (-e_4 - e_7, -e_1 - e_2 + e_3 + e_4 - e_5 - e_6 - e_7 - e_8), (-e_4 - e_7, -e_1 - e_2 + e_3 + e_4 - e_5 - e_6 - e_7 - e_8)\}$. These pairs of roots are all conjugate in E_4 by elements which fix the chosen D_6 root subsystem. Thus, there is a unique D_8-conjugacy class of G_{29} in G_{31}.

Assume $G_{29}, G'_{29} < G_{31}$ $(= C_{E_8}(\rho_4))$. Put $Z = Z(G_{31}) = \{1, \rho_4, \rho_4^2, \rho_4^3\}$ and choose $\phi \in E_8$ such that $\phi G_{29} \phi^{-1} = G'_{29}$. Since $Z \subset G_{29} \cap G'_{29}$, $\phi \rho_4 \phi^{-1} = \rho_4$ or $\phi \rho_4 \phi^{-1} = \rho_4^3$, there are at most two isomorphism classes of G_{29} in G_{31}. In fact using a computer algebra system, it is possible to check that there is only one isomorphism class of G_{29} in G_{31}.

Finally, G_{31} has no imprimitive subgroups for which E_8 is minimal, for suppose $G(m,p,4) < G_{31} < E_8$. Then $G(m,m,4) < G_{31}$ since $G(m,m,4) \leq G(m,p,4)$. But $|G(m,m,4)| = 3(2m)^3$, while $|G_{31}| = 2^{10} \cdot 3 \cdot 5$, so $m \leq 4$. Thus the only possibilities are $G(4,1,4)$, $G(4,2,4)$, and $G(4,4,4)$. Since

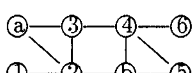

FIG. 1. Carter graph for G_{29}.

G_{31} has no order 4 reflections, $G(4,1,4)$ is eliminated, and by Proposition 3.5, $G(4,4,4) < G(4,2,4) < D_8$.

Remark 3.5. In [8], Shephard described G_{31} as the symmetry group of a four-dimensional complex polytope that has 240 vertices, and he listed the following coordinates for the vertices:

$$\begin{array}{ll} (2i^a,0,0,0) & \text{permuted}, (a = 0,1,2,3) \\ ((1+i)i^a,(1+i)i^b,0,0) & \text{permuted}, (a,b = 0,1,2,3) \\ (i^a,i^b,i^c,i^d) & (a+b+c+d = 0 \ (\operatorname{mod} 2)). \end{array}$$

If Shephard's complex basis is denoted e_1, e_2, e_3, and e_4, then with respect to the *real* ordered basis $\varepsilon_1 = (1+i)e_1$, $\varepsilon_2 = (1-i)e_1, \dots, \varepsilon_7 = (1+i)e_4$, $\varepsilon_8 = (1-i)e_4$, the coordinates are $\pm\varepsilon_i \pm \varepsilon_j$, $i<j$ and $\frac{1}{2}\Sigma \pm \varepsilon_i$ (even number of plus signs), which is the usual realization of the E_8 root system.

B_{2n}

Possible orders of reflections in $G < B_{2n}$ are 2, 3, and 4, so $\Phi = \Phi_2$ or $\Phi = \Phi_3$.

- $\Phi = \Phi_3$. The elements in B_{2n} have characteristic polynomials of the form $\prod(x^{n_i} \pm 1)$, where $\Sigma n_i = 2n$, so $\rho_6 \notin \operatorname{Aut}(\Phi) = B_{2n}$.
- $\Phi = \Phi_2$. There is one conjugacy class in B_{2n} with characteristic polynomial $(x^2 - 2\cos(\frac{2\pi}{4})x + 1)^2$, so the complex reflection group $C_{B_{2n}}(\rho_4)$ is unique.

Since every order two reflection requires four roots from Φ, the rank 3, 4, and 5 primitive reflection groups have too many order 2 reflections to be contained in B_{2n}. Thus, $C_{B_{2n}}(\rho_4)$ is imprimitive, so from [11, Theorem 4.2], it follows that $C_{B_{2n}}(\rho_4) = G(4,1,n)$. Now $G(4,2,n)$ is the subgroup of $G(4,1,n)$ generated by the order 2 reflections and $G(4,4,n)$ is the subgroup generated by the non-diagonal order 2 reflections, so there is only one isomorphism class of each. The argument used to show that $G(4,2,2) < D_4$ can also be used here to prove that $G(4,4,n) < G(4,2,n) < D_{2n}$.

D_{2n}

Since there is one conjugacy class in D_{2n} with characteristic polynomial $(x^2 - 2\cos(\frac{2\pi}{4})x + 1)^n$, the complex reflection group $C_{D_{2n}}(\rho_4)$ is unique, and it follows from the B_{2n} discussion that $C_{D_{2n}}(\rho_4) = G(4,2,n)$ and that the subgroup $G(4,4,n)$ is unique.

TABLE 4
Complex Reflection Subgroups of Real Reflection Groups

W	$G < W$
F_4	$G_{12} \simeq C_{F_4}(\rho_4^{o})$
	$G_8 \simeq C_{F_4}(\rho_4^{i})$
	$G_5 \simeq C_{F_4}(\rho_6^{o})$
B_4	$G(4,1,2) \simeq C_{B_4}(\rho_4)$
D_4	$G(4,2,2) \simeq C_{D_4}(\rho_4)$
	$G(4,4,2) < G(4,2,2)$
	$G_4 \simeq C_{D_4}(\rho_6)$
H_4	$G_{22} \simeq C_{H_4}(\rho_4)$
	$G_{20} \simeq C_{H_4}(\rho_6)$
	$G_5 < G_{20}$
	$G_{16} \simeq C_{H_4}(\rho_{10})$
E_6	$G_{25} \simeq C_{E_6}(\rho_6)$
E_8	$G_{31} \simeq C_{E_8}(\rho_4)$
	$G_{29} < G_{31}$
	$G_{32} \simeq C_{E_8}(\rho_6)$
B_{2n}	$G(4,1,n) \simeq C_{B_{2n}}(\rho_4)$
D_{2n}	$G(4,2,n) \simeq C_{D_{2n}}(\rho_6)$
	$G(4,4,n) < G(4,2,n)$

3.3. *Summary*

Table 4 lists the rank $2n$ $(n > 1)$ irreducible real reflection groups with their truly complex irreducible rank n reflection subgroups. When appropriate, the complex reflection groups are expressed as centralizers.

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